COMPLEX BANACH SPACES WHOSE DUALS ARE L_1-SPACES

BY

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ABSTRACT

We prove that for a complex Banach space A the following properties are equivalent:

i) A^* is isometric to an $L_1(\mu)$-space;

ii) every family of 4 balls in A with the weak intersection property has a non-empty intersection;

iii) every family of 4 balls in A such that any 3 of them have a non-empty intersection, has a non-empty intersection.

1. Introduction

Several recent papers have dealt with complex Banach spaces whose duals are isometric to L_1-spaces. The results obtained are complex analogues of results obtained for real spaces. We will here mention the papers of Effros [2], Hirsberg and Lazar [5], Hustad [7], Lima [9] and Olsen [11].

In [7] Hustad introduced the notion of weak intersection property. He said that a family of balls in a Banach space A has the weak intersection property if their images by every linear functional of norm 1 have a non-empty intersection. He then extended results by Lindenstrauss [10] for real spaces to complex spaces and proved the equivalence of the following statements:

i) A^* is isometric to an $L_1(\mu)$-space;

ii) every finite family of balls in A with the weak intersection property has a non-empty intersection;

iii) every family of 7 balls in A with the weak intersection property has a non-empty intersection.

In the real case, it suffices to consider 4 balls instead of 7 balls in (iii). The aim of this paper is to show that also in the complex case it suffices to consider 4 balls (Theorem 4.1). In Theorem 4.1 we also show, in contrast to the real case, that

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preduals of L_1-spaces are characterized by the following property: every family of 4 balls in A such that any three of them have a non-empty intersection has a point in common. This property is translated into a dual property in A^*, and most of the paper is devoted to the study of this dual property. In Theorem 5.1 we show that this property characterizes complex L_1-spaces.

Throughout the paper we use the following notation. Let A be a Banach space. $B(a,r)$ denotes the closed ball with center a and radius r. The unit ball $B(0,1)$ will also be written A_1. $H^n(A)$ is the space

$$\left\{(x_1,\cdots,x_n)\in A^n : \sum_{i=1}^{n} x_i = 0\right\}$$

with the norm $\|(x_1,\cdots,x_n)\| = \sum_{i=1}^n \|x_i\|$. The convex hull of a set S is denoted $\mathrm{co}(S)$. A convex cone C in A is said to be a *facial cone* if $C = \bigcup_{\lambda\geqq 0}\lambda F$ for some proper face F of A_1, and C is said to be *hereditary* if for all $x \in C$ and $y \in A$ with $\|x\| = \|y\| + \|x-y\|$ we have $y \in C$. Then C is hereditary if and only if C is a union of facial cones [1]. A convex cone C in A is said to have the *Riesz decomposition property* if for all $x_1,\cdots,x_n, y_1,\cdots,y_m \in C$ such that $\sum_{i=1}^n x_i = \sum_{j=1}^m y_j$, there exist $z_{ij}\in C$ such that

$$(*)\qquad x_i = \sum_{j=1}^{m} z_{ij},\quad y_j = \sum_{i=1}^{n} z_{ij},\quad 1\leqq i\leqq n,\quad 1\leqq j\leqq m.$$

If $(*)$ holds with $n = m = 2$, then $(*)$ holds for all n and m. If C is a convex set, $\partial_e C$ denote the set of extreme points of C.

2. The $R_{n,k}$-property

DEFINITION. Let $n > k \geqq 2$ be natural numbers and let A be a real or complex Banach space. We say that A has the *$R_{n,k}$-property* if for all $(x_1,\cdots,x_n)\in H^n(A)$, there exist $(z_{i1},\cdots,z_{in})\in H^n(A)$ where i runs from 1 to $\binom{n}{k}$ such that:

$$\text{(2–1)}\qquad \begin{aligned} &\text{i) } (x_1,\cdots,x_n) = \textstyle\sum_i (z_{i1},\cdots,z_{in});\\ &\text{ii) } \|x_j\| = \textstyle\sum_i \|z_{ij}\| \text{ for all } j;\\ &\text{iii) } (z_{i1},\cdots,z_{in}) \text{ has at most } k \text{ non-zero components for each } i.\end{aligned}$$

The $R_{3,2}$- and $R_{4,2}$-properties were studied in [9] for real Banach spaces. (They were then called R_3 and R_4.) In [9] we proved that a real Banach space A has the $R_{3,2}$-property if and only if A has the 3.2 intersection property, and that

A has the $R_{4,2}$-property if and only if A is isometric to an $L_1(\mu)$-space for some measure μ. From [9, theor. 2.12] it follows that if a Banach space has the $R_{4,2}$-property then it has the $R_{n,2}$-property for all $n \geqq 2$. For complex Banach spaces the $R_{n,2}$-properties are of no interest. It turns out that for complex spaces the $R_{4,3}$-property plays much the same role as the $R_{4,2}$-property does for real spaces. We are now going to prove that if A has the $R_{4,3}$-property then A has the $R_{n,3}$-property for all $n \geqq 4$. First we need a lemma.

LEMMA 2.1. *Let A be a real or complex Banach space with the $R_{4,3}$-property. Then every facial cone of A has the Riesz decomposition property.*

PROOF. Let C be a facial cone of A and let $x_1, x_2, y_1, y_2 \in C$ be such that $x_1 + x_2 = y_1 + y_2$. Then $(x_1, x_2, -y_1, -y_2) \in H^4(A)$. By the $R_{4,3}$-property, there exist $z_{ij} \in A$ such that $(2-1)$ is satisfied. We write it:

$$\begin{aligned}
&(x_1, && x_2, && -y_1, && -y_2)\\
&= (0, && z_{12}, && z_{13}, && z_{14})\\
&+ (z_{21}, && 0, && z_{23}, && z_{24})\\
&+ (z_{31}, && z_{32} && 0, && z_{34})\\
&+ (z_{41}, && z_{42}, && z_{43}, && 0).
\end{aligned}$$

Since C is a facial cone, C is hereditary. Hence by (2–1) (ii), $z_{ij} \in C$ for $i = 1,2,3,4$ and $j = 1,2$ and $-z_{ij} \in C$ for $i = 1,2,3,4$ and $j = 3,4$. Now define elements $u_{ij} \in C$ by

$$u_{11} = z_{41} - z_{23}$$

$$u_{12} = z_{31} - z_{24}$$

$$u_{21} = z_{42} - z_{13}$$

$$u_{22} = z_{32} - z_{14}.$$

Then since $(z_{i1}, \cdots, z_{i4}) \in H^4(A)$ we get

$$\begin{aligned}
u_{11} + u_{12} &= z_{31} + z_{41} - (z_{23} + z_{24})\\
&= z_{31} + z_{41} + z_{21} = x_1
\end{aligned}$$

and also

$$u_{21}+u_{22}=x_2$$

$$u_{11}+u_{21}=y_1$$

$$u_{12}+u_{22}=y_2.$$

Hence C has the Riesz decomposition property. The proof is complete.

THEOREM 2.2. *If A is a real or complex Banach space with the $R_{4,3}$-property, then A has the $R_{n,3}$-property for all $n \geqq 4$.*

PROOF. Since A has the $R_{(n+1),3}$-property if and only if A has the $R_{(n+1),n}$- and the $R_{n,3}$-properties, it suffices to show that if A has the $R_{n,3}$-property, then it has the $R_{(n+1),n}$-property. So assume $n \geqq 4$ and that A has the $R_{n,3}$-property. Let $(x_1,\cdots,x_{n+1}) \in H^{n+1}(A)$. Then $(x_1,\cdots,x_{n-1}, x_n+x_{n+1}) \in H^n(A)$. Clearly A has the $R_{n,(n-1)}$-property, so there exist $z_{ij} \in A$ such that (2–1) is satisfied. We write it:

$$\begin{array}{rlll} & (x_1, & x_2,\cdots,x_{n-1}, & x_n+x_{n+1}) \\ & =(0, & z_{12},\cdots,z_{1(n-1)}, & z_{1n}) \\ & +(z_{21}, & 0,\cdots,z_{2(n-1)}, & z_{2n}) \\ (2\text{–}2) & + & & \\ & \vdots & & \\ & \vdots & & \\ & +(z_{n1}, & z_{n2},\cdots,z_{n(n-1)}, & 0). \end{array}$$

Since $(x_n, x_{n+1}, -z_{1n},\cdots,-z_{(n-3)n}, -z_{(n-2)n}-z_{(n-1)n}) \in H^n(A)$, by the $R_{n,3}$-property there exist $u_{ij} \in A$ such that (2–1) is satisfied. Now we add all elements $(u_{i1},\cdots,u_{in})$ such that $u_{i1}=0$, and then we add all elements $(u_{i1},\cdots,u_{in})$ such that $u_{i1}\neq 0$ and $u_{i2}=0$. Then we get:

$$\begin{array}{rlllllll} & (x_n, & x_{n+1}, & -z_{1n}, & -z_{2n}, & \cdots, & -z_{(n-2)n} & -z_{(n-1)n}) \\ & =(u_{11}, & u_{12}, & u_{13}, & 0, & \cdots, & 0 & \\ & +\cdots & & & & & & \\ (2\text{–}3) & & & & & & & \\ & +(u_{(n-2)1}, & u_{(n-2)2}, & 0, & 0, & \cdots, & u_{(n-2)n} &) \\ & +(u_{(n-1)1}, & 0, & u_{(n-1)3}, & u_{(n-1)4}, & \cdots, & u_{(n-1)n} &) \\ & +(0, & u_{n2}, & u_{n3}, & u_{n4}, & \cdots, & u_{nn} &) \end{array}$$

where (i) and (ii) from (2–1) are satisfied and (iii) from (2–1) is satisfied by the $(n-2)$ first u-lines.

Let C be the facial cone generated by $-(x_n + x_{n+1})$. Then by (2–2) $-z_{in} \in C$ for $i = 1, \cdots, (n-1)$, and by (2–3), $u_{ij} \in C$ for all i and for all $j \geqq 3$. (The property (ii) from (2–1) is satisfied also in (2–2) and (2–3).) By Lemma 2.1 there exist $v_{in}, w_{in} \in C$ such that

$$u_{in} = v_{in} + w_{in} \quad \text{for } i = (n-2), (n-1), n,$$

$$-z_{(n-2)n} = \sum_i v_{in}$$

and

$$-z_{(n-1)n} = \sum_i w_{in}.$$

Now we split the last column in (2–3) and get:

$$\text{(2–4)} \qquad \begin{aligned} &(x_n, \mid x_{n+1}, \quad -z_{1n}, \quad \cdots, \quad -z_{(n-2)n}, \quad -z_{(n-1)n}) \\ &= (u_{11}, \quad u_{12}, \quad u_{13}, \quad \cdots, \quad 0, \quad 0) \\ &+ \cdots \\ &+ (u_{(n-2)1}, \quad u_{(n-2)2}, \quad 0, \quad \cdots, \quad v_{(n-2)n}, \quad w_{(n-2)n}) \\ &+ (u_{(n-1)1}, \quad 0, \quad u_{(n-1)3}, \quad \cdots, \quad v_{(n-1)n}, \quad w_{(n-1)n}) \\ &+ (0, \quad u_{n2}, \quad u_{n3}, \quad \cdots, \quad v_{nn}, \quad w_{nn}). \end{aligned}$$

In (2–4) the properties (i) and (ii) from (2–1) are satisfied and also property (iii) except for the last three lines. The next thing to do is to use the $R_{4,3}$-property to decompose $(u_{(n-2)1}, u_{(n-2)2}, v_{(n-2)n}, w_{(n-2)n}) \in H^4(A)$. Line three from below in (2–4) is then replaced by these four new lines which all satisfy (iii) in (2–1). Then we add all lines with 0 in the first component, and then we add all lines with non-zero first component and 0 in the second component. Hence we get:

$$\text{(2–5)} \qquad \begin{aligned} &(x_n, \quad x_{n+1}, \quad -z_{1n}, \quad -z_{2n}, \quad \cdots, \quad -z_{(n-1)n}) \\ &= (a_{11}, \quad a_{12}, \quad a_{13}, \quad 0, \quad \cdots, \quad 0) \\ &+ \cdots \\ &+ (a_{(n-1)1}, \quad a_{(n-1)2}, \quad 0, \quad 0, \quad \cdots, \quad a_{(n-1)(n+1)}) \\ &+ (b_{11}, \quad 0, \quad b_{13}, \quad b_{14}, \quad \cdots, \quad b_{1(n+1)}) \\ &+ (0, \quad b_{22}, \quad b_{23}, \quad b_{24}, \quad \cdots, \quad b_{2(n+1)}) \end{aligned}$$

where (i) and (ii) from (2–1) are satisfied and also (iii) from (2–1) is satisfied except for the last two lines.

We now split the last column in (2–2) as follows:

$$
\begin{aligned}
&(x_1, && x_2, && \cdots, && x_{n-1}, && x_n, && x_{n+1})\\
&= (0, && z_{12}, && \cdots, && z_{1(n-1)}, && a_{11} - b_{13}, && a_{12} - b_{23})\\
&+ (z_{21}, && 0, && \cdots, && z_{2(n-1)}, && a_{21} - b_{14}, && a_{22} - b_{24})\\
&+ \cdots \\
&+ (z_{(n-1)1}, && z_{(n-1)2}, && \cdots, && 0 && a_{(n-1)1} - b_{1(n+1)}, && a_{(n-1)2} - b_{2(n+1)})\\
&+ (z_{n1}, && z_{n2}, && \cdots, && z_{n(n-1)}, && 0, && 0).
\end{aligned}
\tag{2–6}
$$

Now it only remains to show that (2–6) satisfies (i), (ii) and (iii) (with $k = n$) from (2–1). From (2–5) we get

$$\sum_i (a_{i1} - b_{1(i+2)}) = \sum_i a_{i1} + b_{11} = x_n,$$

and since all $b_{i3}, b_{i4}, \cdots, \in C$ and the norm is additive on C,

$$\sum_i \| a_{i1} - b_{1(i+2)} \| \leqq \sum_i \| a_{i1} \| + \sum_i \| b_{1(i+2)} \|$$

$$= \sum_i \| a_{i1} \| + \left\| \sum_i b_{1(i+2)} \right\| = \sum_i \| a_{i1} \| + \| b_{11} \| = \| x_n \|.$$

Similar formulas hold for x_{n+1}. The proof is complete.

Theorem 2.2 gives a new proof of the following result:

Corollary 2.3. *A Banach space with the $R_{4,2}$-property, has the $R_{n,2}$-property for all $n \geqq 3$.*

In Section 5 we shall show that a complex Banach space has the $R_{4,3}$-property if and only if it is isometric to an $L_1(\mu)$-space.

Remark. We do not know whether Theorem 2.2 can be generalized to other k than 2 and 3. In the proof of Theorem 2.2 we used in an essential way Lemma 2.1. Since by Helly's theorem [4] and Theorem 3.1 every real three-dimensional space has the $R_{n,4}$-property, Lemma 2.1 does not hold for spaces with the $R_{n,4}$-property.

3. Intersection properties

DEFINITION. Let $n > k \geqq 2$ be natural numbers and let A be a real or complex Banach space. We say that A has the *almost n.k. intersection property* (a.$n.k$.I.P.) if for every family $\{B(a_i, r_i)\}_{i=1}^n$ of n balls in A such that for any k of them

$$\bigcap_{j=1}^{k} B(a_{i_j}, r_{i_j}) \neq \emptyset$$

we have

$$\bigcap_{j=1}^{n} B(a_j, r_j + \varepsilon) \neq \emptyset \text{ for all } \varepsilon > 0.$$

The word *almost* is omitted if we can take $\varepsilon = 0$ as well.

THEOREM 3.1. *Let $n > k \geqq 2$ and let A be a real or complex Banach space. A has the a.n.k.I.P. if and only if A^* has the $R_{n,k}$-property.*

PROOF. Let Ω be the set of all subsets of $\{1, 2, \cdots, n\}$ consisting of exactly k different numbers. Then Ω has cardinality $\binom{n}{k}$. For each $w \in \Omega$, let

$$S_w = \{(f_1, \cdots, f_n)\} \in H^n(A^*)_1 : f_j = 0 \text{ for } j \notin w\}.$$

Then each S_w is convex and w^*-compact, and by Theorem 2.10 in [9], we get that A has the a.$n.k$.I.P. if and only if

$$H^n(A^*)_1 = \mathrm{co}\Big(\bigcup_{w \in \Omega} S_w\Big).$$

Assume first that A has the a.$n.k$.I.P., and let $x_1, \cdots, x_n \in A^*$ be such that $\sum_{j=1}^n x_j = 0$. Without loss of generality, we may assume $\sum_{j=1}^n \| x_j \| = 1$, so $(x_1, \cdots, x_n) \in H^n(A^*)_1$. Then there exist $\lambda_w \geqq 0$ with $\sum_{w \in \Omega} \lambda_w = 1$ and $(z_{w1}, \cdots, z_{wn}) \in S_w$ such that

$$(x_1, \cdots, x_n) = \sum_{w \in \Omega} \lambda_w (z_{w1}, \cdots, z_{wn}).$$

Clearly we only have to verify (2–1) (ii): We have

$$\begin{aligned} 1 = \sum_{j=1}^n \| x_j \| &= \sum_{j=1}^n \Big\| \sum_{w \in \Omega} \lambda_w z_{wj} \Big\| \\ &\leqq \sum_{j=1}^n \sum_{w \in \Omega} \lambda_w \| z_{wj} \| = \sum_{w \in \Omega} \Big(\lambda_w \sum_{j=1}^n \| z_{wj} \| \Big) \\ &\leqq \sum_{w \in \Omega} \lambda_w = 1. \end{aligned}$$

Hence

$$\| x_j \| = \sum_{w \in \Omega} \| \lambda_w z_{wj} \|$$

for all j.

The other implication is also straight-forward. If $(x_1, \cdots, x_n) \in H^n(A^*)_1$ with $\Sigma_{j=1}^n \| x_j \| = 1$, let $z_{ij} \in A^*$ $(j = 1, \cdots, n$ and $i = 1, \cdots, \binom{n}{k})$ be as in (2–1), and define

$$\lambda_i = \sum_{j=1}^{n} \| z_{ij} \|.$$

Then $\Sigma_i \lambda_i = 1$ and

$$(x_1, \cdots, x_n) = \sum_i \lambda_i (\lambda_i^{-1} z_{i1}, \cdots, \lambda_i^{-1} z_{in}) \in \mathrm{co}\Big(\bigcup_{w \in \Omega} S_w \Big).$$

The proof is complete.

From Theorem 3.1 and Theorem 2.2 we get:

Corollary 3.2. *If A has the a.4.3.I.P., then A has the a.n.3.I.P. for all $n \geqq 4$.*

Remark. In Section 4 we shall show that for complex Banach spaces, the word *almost* in Corollary 3.2 can be omitted. We do not know if the same is true for real spaces. In [10] Lindenstrauss proved that if A has the 7.3.I.P., then it has the n.3.I.P. for all $n \geqq 4$.

We will need the next lemma in Section 4.

Lemma 3.3. *Assume that A has the a.4.3.I.P. and that $(x_1, x_2, x_3) \in \partial_e H^3(A^*)_1$ with all $x_i \neq 0$. Then $\| x_i \|^{-1} x_i \in \partial_e A_1^*$.*

Proof. Assume $x_1 = y_1 + y_2$ with $\| x_1 \| = \| y_1 \| + \| y_2 \|$. Then by Theorem 3.1 we can find $z_{ij} \in A^*$ such that (2–1) is satisfied:

$$\begin{array}{llll} (y_1, & y_2, & x_2, & x_3) \\ = (0, & z_{12}, & z_{13}, & z_{14}) \\ + (z_{21}, & 0, & z_{23}, & z_{24}) \\ + (z_{31}, & z_{32}, & 0, & z_{34}) \\ + (z_{41}, & z_{42}, & z_{43}, & 0). \end{array}$$

Hence we get

$$
\begin{aligned}
&(x_1, && x_2, && x_3)\\
&=(z_{12}, && z_{13}, && z_{14})\\
&+(z_{21}, && z_{23}, && z_{24})\\
&+(-z_{34}, && 0, && z_{34})\\
&+(-z_{43}, && z_{43}, && 0)
\end{aligned}
$$

where (ii) from (2–1) is still satisfied. This immediately gives us a convex combination in $H^3(A^*)_1$. Hence $z_{34} = z_{43} = 0 = z_{31} = z_{32} = z_{41} = z_{42}$ and $z_{12} = cz_{21}$ or $z_{21} = cz_{12}$ for some constant c. But this gives that both y_1 and y_2 are multiples of x_1. Hence $\|x_1\|^{-1}x_1 \in \partial_e A^*_1$. x_2 and x_3 are treated similarly.

REMARK. Clearly Lemma 3.3 has a natural generalization to spaces with the a.$(n+1).n$.I.P. for $n \geqq 3$.

In Section 4 we will consider complex spaces only. For real Banach spaces the following result of Lindenstrauss [10, theor. 6.3] is an easy consequence of Lemma 3.3.

THEOREM 3.4. *Let A be a closed subspace of $C_{\mathbf{R}}(K)$, for some compact Hausdorff space K, containing the constants. If A has the a.4.3.I.P., then it has the a.n.2.I.P. for all $n \geqq 2$.*

PROOF. Let $(x_1, x_2, x_3) \in \partial_e H^3(A^*)_1$. By theorem 2.10 in [9] it suffices to show that one $x_i = 0$. Assume for contradiction that all $x_i \neq 0$. From Lemma 3.3 it follows that $x_i = \lambda_i \varepsilon_i|_A$ where $|\lambda_i| = \|x_i\|$ and ε_i is a point-measure on K. Since $1 \in A$, we get

$$0 = \sum_{i=1}^{3} \lambda_i \varepsilon_i(1) = \sum_{i=1}^{3} \lambda_i.$$

W.l.o.g. we may assume $\lambda_1 < 0$ and $\lambda_2, \lambda_3 > 0$. The convex sum

$$\frac{x_1}{\lambda_1} = \frac{\lambda_2}{\lambda_2 + \lambda_3}\left(\frac{x_2}{\lambda_2}\right) + \frac{\lambda_3}{\lambda_2 + \lambda_3}\left(\frac{x_3}{\lambda_3}\right)$$

gives us a contradiction to $x_1/\lambda_1 \in \partial_e A^*_1$. The proof is complete.

4. Complex preduals of L-spaces

In this Section A will denote a complex Banach space. We will solve problem 1, and give partial solutions to problem 2 and 3 of Hustad [7]. All solutions are positive.

DEFINITION. A family of balls $\{B(a_i, r_i)\}_{i\in I}$ in a complex (real) Banach space A is said to have the weak intersection property if for any $f \in A_1^*$,

$$\bigcap_{i\in I} B(f(a_i), r_i) \neq \emptyset \text{ in } \mathbf{C}(\mathbf{R}).$$

By Helly's theorem, in real Banach spaces a family of balls has the weak intersection property if and only if they are mutually intersecting, and in complex Banach spaces a family of balls has the weak intersection property if and only if every subfamily of three balls has the weak intersection property.

DEFINITION. Let $n \geqq 3$ be a natural number. We say that a real or complex Banach space A is an *almost* $E(n)$*-space* if for every family $\{B(a_i, r_i)\}_{i=1}^n$ of n balls in A with the weak intersection property, we have

$$\bigcap_{i=1}^{n} B(a_i, r_i + \varepsilon) \neq \emptyset \text{ for all } \varepsilon > 0. \tag{4–1}$$

We say that A is an $E(n)$*-space* if we can take $\varepsilon = 0$ in (4–1).

Theorem 4.1 is the main theorem of this paper.

THEOREM 4.1. *Let A be a complex Banach space. Then the following statements are equivalent*:

i) A^{**} *is a* P_1*-space*;

ii) A^* *is isometric to an* $L_1(\mu)$*-space for some measure* μ;

iii) *if* $\{B(a_i, r_i)\}_{i\in I}$ *is any family of balls in A with the weak intersection property such that the set of centers* $\{a_i\}_{i\in I}$ *is relatively norm-compact, then* $\bigcap_{i\in I} B(a_i, r_i) \neq \emptyset$;

iv) *A is an* $E(n)$*-space for all* $n \geqq 3$;

v) *A is an* $E(4)$*-space*;

vi) *A has the n.3.I.P. for all* $n \geqq 4$;

vii) *A has the* 4.3.*I.P.*

PROOF. The equivalence (i) ⇔ (ii) is due to Hasumi [3] and Sakai [12]. The equivalence (i) ⇔ (iv) is due to Hustad [7]. (See also Hustad [6] and Lima [9].) The following equivalences are trivial:

$$\begin{array}{ccccc} \text{(iii)} & \Rightarrow & \text{(iv)} & \Rightarrow & \text{(v)} \\ & & \Downarrow & & \Downarrow \\ & & \text{(vi)} & \Rightarrow & \text{(vii)}. \end{array}$$

Hence we only have to prove (vii) $\Rightarrow$ (iv) $\Rightarrow$ (iii). These implications follow from Corollary 4.3 and Proposition 4.4 below.

LEMMA 4.2. *Let A be a complex Banach space with the a.4.3.I.P. If $n \geqq 3$ and $(x_1, \cdots, x_n) \in \partial_e H^n(A^*)_1$, then there exist $z_j \in \mathbf{C}$ with $\Sigma_{j=1}^n z_j = 0$ and $y \in \partial_e A_1^*$ such that $x_j = z_j y$ for all j.*

PROOF. By Corollary 3.2, A has the a.n.3.I.P. Let $(x_1, \cdots, x_n) \in \partial_e H^n(A^*)_1$. By theorem 2.10 in [9], $(x_1, \cdots, x_n)$ has at most three components different from 0. Hence it suffices to consider an element $(x_1, x_2, x_3) \in \partial_e H^3(A^*)_1$. If one $x_j = 0$, there is nothing to prove, so assume all $x_j \neq 0$. Let us write $\lambda_j = \| x_j \|$ and $\varepsilon_j = \lambda_j^{-1} x_j$. By Lemma 3.3 all $\varepsilon_j \in \partial_e A_1^*$. We have all $\lambda_j > 0$ and $\Sigma_{j=1}^3 \lambda_j = 1$. Let i denote the imaginary unit. Since $(\lambda_1(1+i)\varepsilon_1, \lambda_1(1-i)\varepsilon_1, 2\lambda_2\varepsilon_2, 2\lambda_3\varepsilon_3) \in H^4(A^*)$ and A^* has the $R_{4,3}$-property by Theorem 3.1, there exist $z_{kj} \in A^*$ such that (2–1) is satisfied.

$$\begin{aligned}
&(\lambda_1(1+i)\varepsilon_1, && \lambda_1(1-i)\varepsilon_1, && 2\lambda_2\varepsilon_2, && 2\lambda_3\varepsilon_3)\\
&=(0, && z_{12}, && z_{13}, && z_{14})\\
&+(z_{21}, && 0, && z_{23}, && z_{24})\\
&+(z_{31}, && z_{32}, && 0, && z_{34})\\
&+(z_{41}, && z_{42}, && z_{43}, && 0).
\end{aligned}$$

At least one $z_{k4} \neq 0$.

Assume $z_{14} \neq 0$. Then since $\varepsilon_3 \in \partial_e A_1^*$, there exists $r > 0$ such that $z_{14} = 2r\lambda_3\varepsilon_3$, and similarly there exist $s \geqq 0$ and $t \geqq 0$ such that

$$\begin{aligned}
0 &= z_{12} + z_{13} + z_{14}\\
&= s\lambda_1(1-i)\varepsilon_1 + t2\lambda_2\varepsilon_2 + r2\lambda_3\varepsilon_3.
\end{aligned}$$

Since

$$0 = 2r\lambda_1\varepsilon_1 + 2r\lambda_2\varepsilon_2 + 2r\lambda_3\varepsilon_3$$

we get

$$\lambda_1(s(1-i)-2r)\varepsilon_1 + 2\lambda_2(t-r)\varepsilon_2 = 0.$$

Now $\lambda_1 \neq 0$ and $s(1-i) \neq 2r$, so $\varepsilon_1 = z\varepsilon_2$ for some $z \in \mathbf{C}$ with $|z| = 1$. Hence also $\varepsilon_3 = \lambda\varepsilon_2$ for some $\lambda \in \mathbf{C}$ with $|\lambda| = 1$, and the conclusion of the lemma follows.

The case $z_{24} \neq 0$ is treated similarly.

Assume $z_{34} \neq 0$. Then there exist $r > 0$ and $s, t \geqq 0$ such that

$$2r\lambda_3\varepsilon_3 = z_{34} = -(z_{31} + z_{32})$$
$$= -(s\lambda_1(1+i) + t\lambda_1(1-i))\varepsilon_1.$$

Hence $\varepsilon_3 = z\varepsilon_1$ and also $\varepsilon_2 = \lambda\varepsilon_1$ for some $z, \lambda \in \mathbf{C}$ with $|z| = |\lambda| = 1$. This concludes the proof of the lemma.

From Lemma 4.2 and [9, theor. 2.14] and [7, prop. 4.8] follows:

COROLLARY 4.3: *If a complex Banach space A has the a.4.3.I.P., then it is an* $E(n)$*-space for all* $n \geqq 3$.

PROPOSITION 4.4. *Assume A is a complex Banach space and that A is an* $E(n)$*-space for all* $n \geqq 3$. *If* $\{B(a_i, r_i)\}_{i\in I}$ *is a family of balls in A with the weak intersection property such that the set of centers* $\{a_i\}_{i\in I}$ *is relatively norm-compact, then* $\bigcap_{i\in I} B(a_i, r_i) \neq \emptyset$.

PROOF. Let $\{B(a_i, r_i)\}_{i\in I}$ be a family of balls in A with the weak intersection property and such that $\{a_i\}_{i\in I}$ is relatively norm-compact. By Lemma 2.2 and the hypothesis in the proposition,

$$\bigcap_{i\in I} B(a_i, r_i + \partial) \neq \emptyset \text{ for all } \partial > 0.$$

Clearly we may assume that all $r_i \leqq K$ for some constant K. Define

$$f(t, \varepsilon) = \sqrt{t^2 + \varepsilon^2} - t.$$

Then

$$f(t, \varepsilon) \geqq \frac{\varepsilon^2}{3K}$$

for small $\varepsilon > 0$ and all $t \in [0, K]$. Choose $\varepsilon > 0$ and define $\partial_n = (\varepsilon/2^n)^2/3K$ for all n. Choose $x_0 \in \bigcap_{i\in I} B(a_i, r_i + \partial_0)$. Now we choose inductively $(x_n)_{n=0}^{\infty}$ in A such that

$$x_{n+1} \in B(x_n, \varepsilon/2^n + \partial_{n+1}) \cap \bigcap_{i\in I} B(a_i, r_i + \partial_{n+1}).$$

This is clearly possible since, when x_n is found, $\{B(x_n, \varepsilon/2^n)\} \cup \{B(a_i, r_i)\}_{i\in I}$ has the weak intersection property by the choice of ∂_n and by lemma 6.4 in [9]. Then $(x_n)_{n=0}^{\infty}$ is a Cauchy-sequence in A and

$$x = \lim_{n\to\infty} x_n \in \bigcap_{i\in I} B(a_i, r_i).$$

This completes the proof of Proposition 4.4 and Theorem 4.1.

REMARK. (vi) and (vii) in Theorem 4.1 solves problem 1 of Hustad [7] to the affirmative. From Corollary 4.3 it follows that if A is an a.$E(4)$-space, then A is an $E(n)$-space for all $n \geqq 3$. This gives a partial solution to problems 2 and 3 of Hustad [7]. We have proved that in some special cases already a.$E(3)$-spaces are $E(n)$-spaces for all n. (This result will appear elsewhere.) This partial result on $E(3)$-spaces and (vii) in Theorem 4.1 is false in the real case. In [7] Hustad proved that if A is an $E(7)$-space, then A is an $E(n)$-space for all $n \geqq 3$.

5. L_1-spaces and the $R_{4,3}$-property

THEOREM 5.1. *Let A be a complex Banach space. Then the following properties are equivalent:*

i) *A is isometric to an $L_1(\mu)$-space;*
ii) *A has the $R_{n,3}$-property for all $n \geqq 4$;*
iii) *A has the $R_{4,3}$-property.*

PROOF.

(iii) ⇒ (i). Assume A has the $R_{4,3}$-property. By theorem 2.1 in [9] it follows that A^* has the 4.3.I.P. Hence by Theorem 4.1, A^* is an $E(n)$-space for all $n \geqq 3$. By w^*-compactness for balls in A^* and a theorem of Hustad [6], A^* is a P_1-space. Hence by results of Hasumi [3] and Sakai [12], A is isometric to an $L_1(\mu)$-space.

(i) ⇒ (ii). Suppose that A is isometric to an $L_1(\mu)$-space. Then by [12], A^* is isometric to a $C(K)$-space, and by [3] and [6] A^* is an $E(n)$-space for all $n \geqq 3$. But then by Theorem 4.1 and Theorem 3.1, A^{**} has the $R_{n,3}$-property for all $n \geqq 4$. By a known result (see also (iii) ⇒ (i)) A^{**} is isometric to an $L_1(\upsilon)$-space, and by [8, §17, theor. 3] A is the range of a contractive projection in A^{**}. Hence A has the $R_{n,3}$-property for all $n \geqq 4$. The proof is complete.

The following result is due to Lindenstrauss and Tzafriri (see [2]).

COROLLARY 5.2. *If A is a complex $L_1(\mu)$-space and P is a projection in A with $\|P\| = 1$, then $P(A)$ is isometric to an $L_1(\upsilon)$-space.*

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